Indian Statistical Institute, Bangalore
B.Math (Hons.) II Year, First Semester
Semestral Examination, Back Paper
Analysis III

Time: 3 hours —— Dec 2011 Instructor: Pl. Muthuramalingam
Maximum marks: 50

2 marks for neatness

1. State the following theorems:

   a) Weirstrass theorem for $C[a,b]$.

   b) Weirstrass M-test for continuous bounded real valued function on a metric space (X,d).

   c) Inverse function theorem.

   d) Implicit function theorem.

   e) Change of variables and the Jacobian formula.

   f) Greens theorem.

   g) Stokes theorem.

   h) Gauss divergence theorem.

   [8× 2=16]

2. Let $f:[a,b)\to R$ be any bounded continuous real valued function.

   i) Let $a\le b_n<b$ and $b_n\to b$ as $n\to\infty$. Define $x_n=\int_a^{b_n} f(t)dt$. Show that $x_1,x_2,x_3,\cdots$ is a cauchy sequence. [2]

   ii) Let $a\le c_n<b$ and $c_n\to b$ as $n\to\infty$. Let $y_n=\int_a^{c_n} f(t)dt$. Show that the sequence $\{x_1,y_1,x_2,y_2,\cdots\}$ is a cauchy sequence. [2]

3. Let $a>0, b>0, k>0$, h real and $ab-h^2>0$. It is known that $\{(x,y) : ax^2+2hxy+by^2\le k\}$ is an elliptic disk. Find its area in terms of $k, ab-h^2$. [4]

4. Prove Gauss divergence theorem for the vector field $\mathbf{F}=(0,0,h)$ and the Tetrahedron V with vertices at O,A,B,C given by $O=(0,0,0), A=(\alpha,0,0), B=(0,b,0), C=(0,0,c)$ with $a,b,c>0$. [4]

5. Let $f:[a,b]\to R$ be any continuous function such that $\int_a^b t^n f(t)dt=0$ for all $n=0,1,2,3,\cdots$. Then show that $\int_a^b [f(t)]^2 dt=0$ and hence $f(t)=0$ for all t. [3]

6. Let $g:[a,b]\to[0,\infty)$ be any continuous function such that $\int_a^b g(t)dt=0$. Show that $g(t)=0$ for all t. [2]

7. Let $f : (0,1) \times (0,1) \to R$ be given by $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$. Show that the iterated integrals for f exist and they are not equal. [4]

8. Let $a, b > 0$ and

$$S = \{(x,y,z) : \quad x^2 + y^2 < a^2 \quad \tfrac{x}{a} + \tfrac{z}{b} = 1\},$$

$$\lceil = \{(x,y,z) : \quad x^2 + y^2 = a^2 \quad \tfrac{x}{a} + \tfrac{z}{b} = 1\}.$$

Note that $\lceil$ is boundary ∂S of S. Find an open set $G \subseteq R^2$ and a $\varphi : G \cup \partial G \to R^3$ so that φ is smooth on G, $1-1$ on G, $\varphi(G) = S$ and $\varphi(\partial G) = \partial S$. [3]

9. Let $f_k : (0,\infty) \times R \times R \to R$ be given by

$$f_k(t,x,y) = e^{-tk^{\cdot 001}} \sin(k^{\cdot 002}x) \cos(k^{\cdot 003}y)$$

.

Define $f : (0,\infty) \times R \times R \to R$ by $f(t,x,y) = \sum_{k=1}^{\infty} f_k(t,x,y)$.

(a) Show that RHS is summable.

(b) Show that f is a continuous function. [4]

10. Find out the volume of $\{(x,y,z) : z \geq 0, x^2 + y^2 + z^2 \leq a^2\}$. [4]